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Exact Values of n-Widths and Optimal Quadratures on Classes of Bounded Analytic and Harmonic Functions

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In this paper we find some exact values of n-widths in the integral metric with the Chebyshev weight function for the classes of functions that are bounded and analytic or harmonic in the interior of the ellipse with foci ± 1 and sum of semiaxes c. We also construct optimal quadrature formulas for these classes.

INTRODUCTION

The Kolmogorov n-width of a subset A of a normed linear space X is defined by

$$d_n(A, X) := \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|,$$

where X_n runs over all n-dimensional subspaces of X. If the infimum is attained by some X_n, then X_n is called an optimal subspace for $d_n(A, X)$.

We will also study the linear n-width defined by

$$\lambda_n(A, X) := \inf_{P_n} \sup_{x \in A} \|x - P_n x\|,$$

where P_n runs over all bounded linear operators mapping X into X whose range has dimension n or less, and the Gel'fand n-width defined by

$$d^n(A, X) := \inf_{X^n} \sup_{x \in A \cap X^n} \|x\|,$$

where the infimum is taken over all subspaces X^n of X of codimension n. In the last definition we assume that $0 \in A$. (The usually considered case is when A is a convex and balanced set.) A detailed bibliography and history of the subject can be found in the book of A. Pinkus [1].

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Let G be a domain in the complex plane. Let $H_\infty(G)$ be the space of bounded analytic functions on G with the norm

$$\|f\|_{H_\infty(G)} := \sup_{z \in G} |f(z)|.$$

The analogous space of harmonic functions we denote by $h_\infty(G)$. We shall write H_∞ and h_∞ if $G = D := \{z \in \mathbb{C} : |z| < 1\}$. Denote by BX the closed unit ball of the normed space X.

Let $E \subset (-1,1)$ be a compact set and let $L_q\,(E, \mu)$ be the Lebesgue space with positive measure μ on E and $1 \leq q \leq \infty$. In Section 1 we obtain the values of the n-widths of Bh_∞ in $L_q(E, \mu)$. We also find the exact value and two different optimal spaces for $d_n(Bh_\infty(\mathscr{E}_c), C[-1,1])$, where $\mathscr{E}_c$ is the interior of the ellipse with foci at the points ± 1 and sum c of its semiaxes.

In section 2 we find the exact values of n-widths of $BH_\infty(\mathscr{E}_c)$ and $Bh_\infty(\mathscr{E}_c)$ in $L_q\,([-1,1], \mu)$ for $d\mu(x) = dx/\sqrt{1-x^2}$, $1 \leq q < \infty$. To obtain this result we solve some minimization problem with Blashke products. The solution of this problem allows us to construct optimal quadrature formulas for the classes $BH_\infty(\mathscr{E}_c)$ and $Bh_\infty(\mathscr{E}_c)$ and to improve the results of Refs. [2, 3], where we proved that these formulas are optimal for sufficiently large c.

1. n-WIDTHS OF HARMONIC FUNCTIONS IN h_∞

A Blashke product of degree n is a function of the form

$$B(z) = \sigma \prod_{j=1}^{m} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \qquad |\alpha_j| < 1, \qquad j = 1, \ldots, m, \qquad |\sigma| = 1.$$

Denote by $\mathscr{B}_n$ the set of all Blashke products of degree n or less and by $\mathscr{B}_n^0$ the ones with $\alpha_j \in (-1,1)$, $\sigma = \pm 1$. Let E be a compact subset of D and μ a positive measure on E such that $\mu(E) < \infty$. Denote by $L_q :=$ $L_q(E, \mu)$ the Lebesgue space of functions on E with the usual norm $\|\cdot\|_q$.

It was proved by Fisher and Micchelli [4] that

$$d_n(BH_\infty, L_q) = \lambda_n(BH_\infty, L_q) = d^n(BH_\infty, L_q) = \inf_{B \in \mathscr{B}_n} \|B\|_q. \tag{1}$$

We obtain a similar result for the class Bh_∞.

THEOREM 1. *Let $E \subset (-1,1)$. Then for all $1 \le q \le \infty$*

$$d_n(Bh_\infty, L_q) = \lambda_n(Bh_\infty, L_q) = d^n(Bh_\infty, L_q) = (4/\pi) \inf_{B \in \mathscr{B}_n^0} \|\arctan B\|_q.$$

Proof. We use the scheme of proof from [4]. Let $x_1, \dots, x_n$ be any points in $(-1,1)$ and let $B(z)$ be the Blashke product with the zeros at these points. In [3] an optimal recovery method was obtained for the functional $u(x)$, $u \in Bh_\infty$, $x \in (-1,1)$, based on the information $u(x_1), \dots, u(x_n)$. It was also proved that the error of this method is equal to $(4/\pi)|\arctan B(x)|$. Thus there exist functions $g_1, \dots, g_n \in C(E)$ such that for all $u \in Bh_\infty$ and all $x \in (-1,1)$

$$\left| u(x) - \sum_{j=1}^{n} g_j(x) u(x_j) \right| \le \frac{4}{\pi} |\arctan B(x)|,$$

where successive derivatives of u at x_j through order $r-1$ will appear if some x_j coincide with order r. Hence

$$d_n(Bh_\infty, L_q) \le \lambda_n(Bh_\infty, L_q) \le \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^0} \|\arctan B\|_q. \tag{2}$$

To obtain the reverse inequality we use the Borsuk Theorem (see, for example, [1]). Fix points $x_0, \dots, x_n \in (-1,1)$. Let $y = (y_0, \dots, y_n) \in S^n := \{y \in \mathbb{R}^{n+1}: \sum_{j=0}^n y_j^2 = 1\}$. Set

$$\rho(y) := \inf_{\substack{f \in H_\infty \\ f(x_j) = y_j,\, j = 0, \dots, n}} \|f\|_{H_\infty}.$$

According to the classical Pick–Nevanlinna Theorem there is a unique Blashke product $B \in \mathscr{B}_n$ such that

$$\rho(y) B(x_j) = y_j, \qquad j = 0, \dots, n. \tag{3}$$

Since $\overline{B(\bar{z})}$ satisfies the Eqs. (3), it follows that B is real on the real axis. Denote by $\mathscr{B}_n^R$ the set of all Blashke products $B \in \mathscr{B}_n$ which are real on the real axis and by T the mapping

$$(Ty)(z) := \frac{4}{\pi} \arctan B(z),$$

where B satisfies (3). The function

$$w = \frac{4}{\pi}\arctan z$$

is a conformal mapping of D on the strip $|\mathrm{Re}\, w| < 1$. Therefore Re $Ty \in Bh_\infty$ for every $y \in S^n$. The continuity of $\rho(y)$ implies that T is a continuous and odd map of S^n into L_q.

Let $1 < q < \infty$. Suppose that $X_n = \mathrm{span}\{f_1, \dots, f_n\}$ is an n-dimensional subspace of L_q. For each $f \in L_q$ let $c_1(f), \dots, c_n(f)$ be the coefficients of $f_1, \dots, f_n$, respectively, in the best approximation to f from X_n. The mapping $Sf := (c_1(f), \dots, c_n(f))$ is a continuous odd mapping of L_q into $\mathbb{R}^n$. Thus $S \circ T$ is an odd, continuous map of S^n into $\mathbb{R}^n$. By the Borsuk Theorem there exists a $y^* \in S^n$ for which $c_j(Ty^*) = 0$, $j = 1, \dots, n$. Hence

$$\sup_{u \in Bh_\infty} \inf_{v \in X_n} \|u - v\|_q \ge \inf_{v \in X_n} \|Ty^* - v\|_q = \|Ty^*\|_q$$
$$\ge \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^R} \|\arctan B\|_q.$$

Since X_n is arbitrary we have

$$d_n(Bh_\infty, L_q) \ge \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^R} \|\arctan B\|_q.$$

For every $\alpha + i\beta \in D$ and $x \in (-1, 1)$

$$\left| \frac{x - \alpha - i\beta}{1 - (\alpha - i\beta)x} \right| \ge \left| \frac{x - \alpha}{1 - \alpha x} \right|. \tag{4}$$

Thus

$$d_n(Bh_\infty, L_q) \ge \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^0} \|\arctan B\|_q.$$

The cases $q = 1, \infty$ are established by passing to the limit as either $q \searrow 1$ or $q \nearrow \infty$.

Now consider the case of d^n. Let X^n be any subspace of L_q of codimension n. Thus

$$X^n = \{u \in L_q : \langle f_j', u \rangle = 0, j = 1, \dots, n\}$$

for some linearly independent and continuous functionals f_j' on L_q.

Denote by $T'\colon S^n \to \mathbb{R}^n$ the mapping

$$T'y := (\langle f'_1, Ty\rangle, \dots, \langle f'_n, Ty\rangle).$$

T' is an odd and continuous map. By the Borsuk Theorem there exists a $y^* \in S^n$ for which $T'y^* = 0$. Since $Ty^* \in Bh_\infty$ we have

$$\sup_{\substack{u \in Bh_\infty \\ \langle f'_j, u\rangle = 0,\, j=1,\dots,n}} \|u\|_q \geq \|Ty^*\|_q \geq \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^R} \|\arctan B\|_q$$

$$= \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^0} \|\arctan B\|_q.$$

As X_n is arbitrary we find

$$d^n(Bh_\infty, L_q) \geq \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^0} \|\arctan B\|_q.$$

The reverse inequality follows from the well-known inequality (see, for example, [1])

$$\lambda_n(A, X) \geq d^n(A, X)$$

and (2). The theorem is proved. ▮

We shall use the standard notation for the Jacobi elliptic function $w = \operatorname{sn}(z,k)$, which is defined from the equation

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Besides that we shall deal with the elliptic functions

$$\operatorname{cn}(z,k) := \sqrt{1 - \operatorname{sn}^2(z,k)}, \qquad \operatorname{dn}(z,k) := \sqrt{1 - k^2\operatorname{sn}^2(z,k)}$$

($\operatorname{cn}(0,k) = \operatorname{dn}(0,\mathrm{k}) = 1$) and complete elliptic integrals of the first kind with moduli k and $k' = \sqrt{1-k^2}$,

$$K := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad K' := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

From [5] it follows that for every $k \in (0,1)$

$$\inf_{B \in \mathscr{B}_n} \sup_{x \in [-\sqrt{k}, \sqrt{k}]} |B(x)| = \sup_{x \in [-\sqrt{k}, \sqrt{k}]} |Z_n(x)| = 2h^{n/4} \frac{\sum_{m=0}^{\infty} h^{nm(m+1)}}{1 + 2\sum_{m=1}^{\infty} h^{nm^2}},$$

where $h = e^{-\pi K'/K}$,

$$Z_n(z) := \prod_{j=1}^{n} \frac{z - z_j^0}{1 - z_j^0 z},$$

$$z_j^0 := \sqrt{k}\,\mathrm{sn}\left[\left(\frac{2j-1}{n} - 1\right)K, k\right], \qquad j = 1, \ldots, n.$$

As the function arctan x is monotone we have from Theorem 1

$$\begin{aligned} d_n\left(Bh_\infty, C\left[-\sqrt{k}, \sqrt{k}\right]\right) &= \lambda_n\left(Bh_\infty, C\left[-\sqrt{k}, \sqrt{k}\right]\right) \\ &= d^n\left(Bh_\infty, C\left[-\sqrt{k}, \sqrt{k}\right]\right) \\ &= \frac{4}{\pi}\arctan\left[2h^{n/4} \frac{\sum_{m=0}^{\infty} h^{nm(m+1)}}{1 + 2\sum_{m=1}^{\infty} h^{nm^2}}\right]. \end{aligned} \tag{5}$$

To rewrite the right hand side of (5) we need the following lemma.

LEMMA 1. *For all* $h \in (0,1)$

$$\frac{4}{\pi}\arctan\left[2h \frac{\sum_{m=0}^{\infty} h^{4m(m+1)}}{1 + 2\sum_{m=1}^{\infty} h^{4m^2}}\right] = \frac{8}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{h^{2m+1}}{1 + h^{2(2m+1)}}.$$

Proof. Determine k by the equation

$$e^{-\pi K'/K} = h.$$

Then for real x (see [6])

$$\operatorname{dn}\left(\frac{Kx}{\pi}, k\right) = \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{m=1}^{\infty}\frac{h^m}{1+h^{2m}}\cos mx.$$

Hence

$$\frac{4K}{\pi^2}\int_0^{\pi/2}\operatorname{dn}\left(\frac{Kx}{\pi}, k\right)dx = 1 + \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\frac{h^{2m+1}}{1+h^{2(2m+1)}}. \tag{6}$$

It is easy to obtain that

$$\int \operatorname{dn}(t,k)\,dt = \arctan\frac{\operatorname{sn}(t,k)}{\operatorname{cn}(t,k)} + C.$$

Using the well-known equations from the theory of elliptic functions (see, for example, [6])

$$\operatorname{sn}(K/2,k) = \frac{1}{\sqrt{1+k'}}, \qquad \operatorname{cn}(K/2,k) = \frac{\sqrt{k'}}{\sqrt{1+k'}},$$

$$\sqrt{k'} = \frac{1 + 2\sum_{m=1}^{\infty}(-1)^m h^{m^2}}{1 + 2\sum_{m=1}^{\infty} h^{m^2}},$$

we have by (6)

$$\begin{aligned}
&\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\frac{h^{2m+1}}{1+h^{2(2m+1)}} \\
&\quad = \frac{4}{\pi}\int_0^{K/2}\operatorname{dn}(t,k)\,dt - 1 \\
&\quad = \frac{4}{\pi}\arctan\frac{\operatorname{sn}(x,k)}{\operatorname{cn}(x,k)}\bigg|_0^{K/2} - 1 = \frac{4}{\pi}\left(\arctan\frac{1}{\sqrt{k'}} - \arctan 1\right) \\
&\quad = \frac{4}{\pi}\arctan\frac{1-\sqrt{k'}}{1+\sqrt{k'}} = \frac{4}{\pi}\arctan\left[2h\frac{\sum_{m=0}^{\infty} h^{4m(m+1)}}{1 + 2\sum_{m=1}^{\infty} h^{4m^2}}\right]. \quad \blacksquare
\end{aligned}$$

The function

$$\phi(w) := \sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}\arcsin w, k\right) \qquad (7)$$

maps $\mathscr{E}_c$ conformally on the unit disk D, and carries the interval $[-1, 1]$ to the interval $[-\sqrt{k}, \sqrt{k}]$, where k satisfies

$$\frac{K'}{K} = \frac{4}{\pi}\log c. \qquad (8)$$

Note that the map ϕ carries the Chebyshev points

$$x_j^0 = \cos\frac{2j-1}{2n}\pi, \qquad j = 1,\ldots,n, \qquad (9)$$

to z_j^0. We obtain the following corollary by using this map, (5) and Lemma 1.

COROLLARY 1. *For all* $c > 1$

$$\begin{aligned} d_n(Bh_\infty(\mathscr{E}_c), C[-1,1]) &= \lambda_n(Bh_\infty(\mathscr{E}_c), C[-1,1]) \\ &= d^n(Bh_\infty(\mathscr{E}_c), C[-1,1]) \\ &= \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\,\frac{c^{-(2m+1)n}}{1+c^{-2(2m+1)n}}. \end{aligned}$$

Denote by $A_0(\mathscr{E}_c)$ the class of functions f which are analytic in $\mathscr{E}_c$, real on the real axis and satisfy

$$|\mathrm{Re} f(z)| \le 1, \qquad z \in \mathscr{E}_c.$$

It was proved by N. I. Akhiezer [7] that

$$\begin{aligned} E_n(A_0(\mathscr{E}_c)) &:= \sup_{f\in A_0(\mathscr{E}_c)}\ \inf_{p\in\mathscr{P}_{n-1}} \|f-p\|_{C[-1,1]} \\ &= \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\,\frac{c^{-(2m+1)n}}{1+c^{-2(2m+1)n}}, \end{aligned} \qquad (10)$$

where $\mathscr{P}_{n-1}$ is the set of all polynomials of degree $n-1$ or less.

It is easy to show that the restrictions on the real axis of $A_0(\mathscr{E}_c)$ and $Bh_\infty(\mathscr{E}_c)$ coincide. Thus Eq. (10) is valid for $Bh_\infty(\mathscr{E}_c)$ and the space $\mathscr{P}_{n-1}$ is optimal subspace for $d_n(Bh_\infty(\mathscr{E}_c), C[-1,1])$. By proving Theorem 1 we see that the space

$$X_n := \operatorname{span}\{g_1(\phi(w)), \ldots, g_n(\phi(w))\},$$

where the functions $g_1, \ldots, g_n$ are determined by the points z_j^0, is also an optimal subspace. Akhiezer's result was brought to the author's attention by to V. M. Tikhomirov. He also conjectured that there is a sequence of optimal subspaces, like in the case of smooth functions (see [8]).

2. Exact Values of n-Widths in L_q and Optimal Quadrature Formulas

We first formulate a generalization of one result obtained by A. Pinkus [9] (see also [1, p. 174]). Let $h(t)$ be a piecewise continuous, 2π-periodic function. Denote by $S_c(h)$ the number of sign changes of h. For a real, continuous, 2π-periodic function k set

$$(k * h)(x) := \frac{1}{2\pi}\int_0^{2\pi} k(x-t)h(t)\,dt.$$

The kernel k is nondegenerate cyclic variation diminishing (NCVD) if $S_c(k * h) \le S_c(h)$ for all h, and

$$\dim \operatorname{span}\{k(x_1 - \cdot), \ldots, k(x_n - \cdot)\} = n$$

for every choice of $0 \le x_1 < \cdots < x_n < 2\pi$ and all n. The kernel k is said to be strictly sign consistent of order $2l + 1$ (SSC_{2l+1}) if

$$\sigma \det\big(k(x_j - y_m)\big)_{j,m=1}^{2l+1} > 0$$

whenever $0 \le x_1 < \cdots < x_{2l+1} < 2\pi$, $0 \le y_1 < \cdots < y_{2l+1} < 2\pi$, and $\sigma = 1$ or -1.

Set

$$\Lambda_{2m} := \{\xi : \xi = (\xi_1, \ldots, \xi_{2m}), 0 \le \xi_1 \le \cdots \le \xi_{2m} < 2\pi\}.$$

For each $\xi \in \Lambda_{2m}$ we define

$$h_\xi(t) := (-1)^j, \qquad t \in [\xi_{j-1}, \xi_j), \qquad j = 1, \ldots, 2m+1,$$

where $\xi_0 := 0$, $\xi_{2m+1} := 2\pi$. Denote by $h_m(t)$ the function h_ξ for $\xi_j = (j-1)\pi/m$, $j = 1, \ldots, 2m$.

THEOREM 2. *Let k be an NCVD kernel and φ a nonnegative function defined on* $[0, C]$, *where*

$$C := \frac{1}{2\pi}\int_0^{2\pi}|k(t)|\,dt.$$

Suppose that φ' is an nonnegative, continuous, and strictly increasing function. Then

$$\inf_{\xi\in\Lambda_{2n}}\int_0^{2\pi}\varphi\big(|(k*h_\xi)(t)|\big)\,dt = \int_0^{2\pi}\varphi\big(|(k*h_n)(t)|\big)\,dt.$$

Furthermore, if k is SSC_{2l+1}, $l = 0, 1, \ldots, n$, and the infimum is attained by $\xi^* \epsilon \Lambda_{2n}$, then $\xi^*_{j+1} - \xi^*_j = \pi/n$, $j = 1, \ldots, 2n-1$.

This theorem was proved by A. Pinkus for $\varphi(x) = x^q$, $1 \le q < \infty$. The general case is proven in a similar way. To count sign changes we only need to use the equation

$$\operatorname{sign}(a+b) = \operatorname{sign}(\varphi'(|a|)\operatorname{sign} a + \varphi'(|b|)\operatorname{sign} b)$$

instead of

$$\operatorname{sign}(a+b) = \operatorname{sign}\big(|a|^{q-1}\operatorname{sign} a + |b|^{q-1}\operatorname{sign} b\big), \qquad 1 < q < \infty.$$

Set $D_H := \{z \in \mathbb{C}: |\operatorname{Im} z| < H\}$. Denote by A_H the class of all functions analytic in D_H, real and 2π-periodic on the real axis which satisfy

$$|\operatorname{Re} f(z)| \le 1, \qquad z \in D_H.$$

Each function $f \in A_H$ has the representation

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}K_H(z-t)\operatorname{Re} f(t+iH)\,dt,$$

where

$$K_H(z) = 1 + 2\sum_{j=1}^{\infty}\frac{\cos jz}{\cos jH}$$

and K_H is NCVD on $[0, 2\pi)$ (see [1]). Moreover, it was proved by W. Forst [10] that K_H is SSC_{2l+1} for all $l = 0, 1, \ldots$.

By Theorem 2 with $k = K_H$ we solve some extremal problems which allow us to obtain exact values of n-widths and to construct optimal quadratures for $BH_\infty(\mathscr{E}_c)$ and $Bh_\infty(\mathscr{E}_c)$.

For $k \in (0,1)$ set

$$d\nu_k(z) := \frac{\pi}{2K} \frac{dz}{\sqrt{(k-z^2)(1-kz^2)}}.$$

THEOREM 3. *Let φ be a function defined on* $[0,1]$ *which satisfies the assumptions of Theorem* 2. *Then for all* $k \in (0,1)$

$$\inf_{B \in \mathscr{B}_n^0} \int_{-\sqrt{k}}^{\sqrt{k}} \varphi\left(\frac{4}{\pi}\arctan|B(z)|\right) d\nu_k(z) = \frac{\pi}{\Lambda} \int_0^1 \frac{\varphi\left(\frac{4}{\pi}\arctan(\sqrt{\lambda}\, t)\right) dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}, \tag{11}$$

where

$$\lambda = 4h^{n/2}\left(\frac{\sum_{m=0}^{\infty} h^{nm(m+1)}}{1 + 2\sum_{m=1}^{\infty} h^{nm^2}}\right)^2, \qquad h = e^{-\pi K'/K}, \tag{12}$$

and Λ is the complete elliptic integral of the first kind with modulus λ. Moreover, the functions $\pm Z_n$ are the only functions for which the infimum is attained.

Proof. Let $B \in \mathscr{B}_n^0$. Set

$$f(t) := \frac{4}{\pi}\arctan B\left(\sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}t, k\right)\right).$$

From the properties of the elliptic function $\mathrm{sn}(x,k)$ it follows that $f \in A_H$, where $H = \pi K'/(4K)$. For $z = e^{i\theta}$ we have

$$\begin{aligned}\mathrm{Re}\,B(z) &= \sigma\,\mathrm{Re}\prod_{j=1}^{n}\frac{z-x_j}{1-x_j z} = \sigma\prod_{j=1}^{n}\frac{1}{|1-x_j z|^2}\,\mathrm{Re}\prod_{j=1}^{n}\frac{(z-x_j)^2}{z}\\ &= \sigma\prod_{j=1}^{n}\frac{1}{|1-x_j z|^2}\sum_{j=0}^{n} c_j\cos j\theta,\end{aligned}$$

where $c_j \in \mathbb{R}$, $\sigma = 1$ or -1. Thus the function $\mathrm{Re}\,B(e^{i\theta})$ has at most $2n$

zeros in $(0, 2\pi)$. As t runs from 0 to 2π the point

$$z = \sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}(t + iH), k\right)$$

makes one rotation around the unit circle. Since for all $|z| = 1$ and $z \neq \pm i$

$$\mathrm{Re}\left(\frac{4}{\pi}\arctan z\right) = \mathrm{sign}\,\mathrm{Re}\, z,$$

we have for almost all $t \in [0, 2\pi]$

$$\mathrm{Re}\, f(t + iH) = \mathrm{sign}\,\mathrm{Re}\, B\left(\sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}(t + iH), k\right)\right). \qquad (13)$$

Consequently there exists a $\xi \in \Lambda_{2n}$ for which

$$|f(t)| = |(K_H * h_\xi)(t)|.$$

By using the first fundamental transformation of degree n (see [6]) we can find

$$Z_n\left(\sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}t, k\right)\right) = \begin{cases} (-1)^m\sqrt{\lambda}\,\mathrm{sn}\left(\dfrac{2n\Lambda}{\pi}t + \Lambda, \lambda\right), & n = 2m, \\ (-1)^m\sqrt{\lambda}\,\mathrm{sn}\left(\dfrac{2n\Lambda}{\pi}t, \lambda\right), & n = 2m + 1, \end{cases}$$

where λ is determined by the equation

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}$$

(Λ' is the complete elliptic integral of the first kind with modulus $\lambda' = \sqrt{1 - \lambda^2}$). From the standard equation

$$\sqrt{\lambda} = 2h_1^{1/4}\frac{\sum_{m=0}^{\infty} h_1^{m(m+1)}}{1 + 2\sum_{m=1}^{\infty} h_1^{m^2}}, \qquad h_1 = e^{-\pi\Lambda'/\Lambda},$$

in the theory of elliptic functions, we obtain (12). Set

$$f_n(t) := \frac{4}{\pi}\arctan\left[Z_n\left(\sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}t, k\right)\right)\right].$$

Let $n = 2m + 1$. Then from the equation

$$\operatorname{Re}\,\mathrm{sn}(w + i\Lambda'/2, \lambda) = \frac{(1+\lambda)\mathrm{sn}(w, \lambda)}{1 + \lambda^2\mathrm{sn}^2(w, \lambda)},$$

and (13) it follows

$$\operatorname{Re} f_n(t + iH) = (-1)^m \operatorname{sign}\mathrm{sn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right).$$

Hence

$$f_n(t) = (-1)^m (K_H * h_n)(t).$$

It can be proved similarly that for $n = 2m$

$$f_n(t) = (-1)^m (K_H * h_n)\left(t + \frac{\pi}{2n}\right).$$

Thus

$$(K_H * h_n)(t) = \frac{4}{\pi}\arctan\left[\sqrt{\lambda}\,\mathrm{sn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right)\right]. \tag{14}$$

Now from Theorem 2 we have

$$\inf_{B \in \mathscr{B}_n^0} \int_0^{2\pi} \varphi\left(\frac{4}{\pi}\arctan\left|B\left(\sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}t, k\right)\right)\right|\right) dt$$
$$\geq \inf_{\xi \in \Lambda_{2n}} \int_0^{2\pi} \varphi(|(K_H * h_\xi)(t)|)\, dt = \int_0^{2\pi} \varphi(|(K_H * h_n)(t)|)\, dt$$
$$= \int_0^{2\pi} \varphi(|f_n(t)|)\, dt.$$

In view of the equation $\operatorname{sn}(2K - w, k) = \operatorname{sn}(w, k)$ we will have

$$\inf_{B \in \mathscr{B}_n^0} \int_{-\pi/2}^{\pi/2} \varphi\left(\frac{4}{\pi}\arctan\left|B\left(\sqrt{k}\operatorname{sn}\left(\frac{2K}{\pi}t, k\right)\right)\right|\right) dt$$

$$= \int_{-\pi/2}^{\pi/2} \varphi(|(K_H * h_n)(t)|)\, dt$$

$$= \frac{\pi}{\Lambda}\int_0^{\Lambda} \varphi\left(\frac{4}{\pi}\arctan(\sqrt{\lambda}\operatorname{sn}(z, \lambda))\right) dz.$$

Making the change of variables

$$z = \sqrt{k}\operatorname{sn}\left(\frac{2K}{\pi}t, k\right), \tag{15}$$

in the first integral and $t = \operatorname{sn}(z, \lambda)$ in the last one, we obtain (11).

If the infimum in (11) is attained by any $B^* \in \mathscr{B}_n^0$ then from Theorem 2 there exists an $\alpha \in [0, \pi/n)$ and $\sigma = 1$ or -1 for which

$$\frac{4}{\pi}\arctan B^*\left(\sqrt{k}\operatorname{sn}\left(\frac{2K}{\pi}t, k\right)\right) = \sigma(K_H * h_n)(t + \alpha)$$

$$= \sigma\frac{4}{\pi}\arctan\left[\sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}(t + \alpha), \lambda\right)\right].$$

Thus

$$B^*\left(\sqrt{k}\operatorname{sn}\left(\frac{2K}{\pi}t, k\right)\right) = \sigma\sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}(t + \alpha), \lambda\right).$$

In view of the formula for $\operatorname{sn}(u + w, \lambda)$ we have

$$B^*\left(\sqrt{k}\operatorname{sn}\left(\frac{2K}{\pi}t, k\right)\right)$$

$$= \frac{a\operatorname{sn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right) + b\operatorname{cn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right)\operatorname{dn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right)}{1 - c\operatorname{sn}^2\left(\frac{2n\Lambda}{\pi}t, \lambda\right)}, \tag{16}$$

where

$$b = \sigma\sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}\alpha, \lambda\right)$$

(the numbers a and c are irrelevant). Let $n = 2m + 1$. If we make the change of variable (15), the left hand side of (16) and sn $((2n\Lambda/\pi)t, \lambda)$ become rational functions. On the other hand, it is not difficult to show that

$$\operatorname{cn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right)\operatorname{dn}\left(\frac{2n\Lambda}{\pi}t, \lambda\right)$$

is not a rational function (as a function of z). Therefore $b = 0$ and consequently $\alpha = 0$. This means that $B^* = Z_n$ or $-Z_n$. The case $n = 2m$ can be considered similarly. The theorem is proved. ∎

Set

$$I_{q0}(\lambda) := \int_0^1 \frac{t^q\,dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}},$$

$$I_{q1}(\lambda) := \int_0^1 \frac{\arctan^q(\sqrt{\lambda}\,t)\,dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}},$$

$$I_{q2}(\lambda) := \int_0^1 \frac{\arctan(\sqrt{\lambda}\,t)^q\,dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}.$$

Considering in Theorem 3 the functions

$$\tan^q\frac{\pi}{4}x, \qquad \left(\frac{\pi}{4}x\right)^q, \qquad \arctan\left(\tan^q\frac{\pi}{4}x\right), \qquad 1 \le q < \infty,$$

as φ, we have

COROLLARY 2. *For all $k \in (0,1)$ and $1 \le q < \infty$*

$$\inf_{B \in \mathscr{B}_n^0} \int_{-\sqrt{k}}^{\sqrt{k}} |B(z)|^q\,d\nu_k(z) = \frac{\pi}{\Lambda}\lambda^{q/2} I_{q0}(\lambda),$$

$$\inf_{B \in \mathscr{B}_n^0} \int_{-\sqrt{k}}^{\sqrt{k}} \arctan^q |B(z)|\,d\nu_k(z) = \frac{\pi}{\Lambda} I_{q1}(\lambda),$$

$$\inf_{B \in \mathscr{B}_n^0} \int_{-\sqrt{k}}^{\sqrt{k}} \arctan |B(z)|^q\,d\nu_k(z) = \frac{\pi}{\Lambda} I_{q2}(\lambda).$$

Furthermore, in every case the infimum is attained only by $\pm Z_n$.

Now we can prove the following result.

THEOREM 4. *Let* $L_q := L_q([-1,1], \mu)$, $d\mu(x) = dx/\sqrt{1-x^2}$. *For all* $1 \le q < \infty$ *and* $c > 1$

$$\begin{aligned} &d_n(BH_\infty(\mathscr{E}_c), L_q) \\ &\quad = \lambda_n(BH_\infty(\mathscr{E}_c), L_q) = d^n(BH_\infty(\mathscr{E}_c), L_q) \\ &\quad = \sqrt{\lambda}\left(\frac{\pi}{\Lambda} I_{q0}(\lambda)\right)^{1/q} = 2\left(\sqrt{\pi}\,\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)}\right)^{1/q} c^{-n} + O(c^{-5n}), \end{aligned}$$

$$\begin{aligned} &d_n(Bh_\infty(\mathscr{E}_c), L_q) \\ &\quad = \lambda_n(Bh_\infty(\mathscr{E}_c), L_q) = d^n(Bh_\infty(\mathscr{E}_c), L_q) \\ &\quad = \frac{4}{\pi}\left(\frac{\pi}{\Lambda} I_{q1}(\lambda)\right)^{1/q} = \frac{8}{\pi}\left(\sqrt{\pi}\,\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)}\right)^{1/q} c^{-n} + O(c^{-5n}), \end{aligned}$$

where Λ *is the complete elliptic integral of the first kind with modulus*

$$\lambda = 4c^{-2n}\left[\frac{\sum_{m=0}^{\infty} c^{-4nm(m+1)}}{1 + 2\sum_{m=1}^{\infty} c^{-4nm^2}}\right]^2. \tag{17}$$

Proof. Let $z = \phi(w)$ be a conformal mapping of $\mathscr{E}_c$ on the unit disk D determined by (7). It is easy to show that

$$d\nu_k(z) = \frac{dw}{\sqrt{1-w^2}}.$$

Therefore by the mapping ϕ, the original problem reduces to one of finding the exact values of the n-widths of BH_∞ and Bh_∞ in $L_q([-\sqrt{k}, \sqrt{k}], \nu_k)$. From (1) we have

$$d_n\left(BH_\infty, L_q\left(\left[-\sqrt{k}, \sqrt{k}\right], \nu_k\right)\right) = \inf_{B \in \mathscr{B}_n}\left(\int_{-\sqrt{k}}^{\sqrt{k}} |B(z)|^q \, d\nu_k(z)\right)^{1/q}.$$

It follows from (4) that we can change $\mathscr{B}_n$ by $\mathscr{B}_n^0$ in the last equation. Thus by Corollary 2

$$d_n\big(BH_\infty(\mathscr{E}_c), L_q\big) = \sqrt{\lambda}\left(\frac{\pi}{\Lambda} I_{q0}(\lambda)\right)^{1/q}.$$

For the class $Bh_\infty(\mathscr{E}_c)$ from Theorem 1 and Corollary 2 we obtain

$$\begin{aligned} d_n\big(Bh_\infty(\mathscr{E}_c), L_q\big) &= \frac{4}{\pi} \inf_{B \in \mathscr{B}_n^0} \left(\int_{-\sqrt{k}}^{\sqrt{k}} \arctan^q |B(z)|\, d\nu_k(z)\right)^{1/q} \\ &= \frac{4}{\pi}\left(\frac{\pi}{\Lambda} I_{q1}(\lambda)\right)^{1/q}. \end{aligned}$$

The asymptotic equations follow from (17) and the well-known equations

$$\int_0^1 x^q (1-x^2)^{-1/2}\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)},$$

$$\Lambda = \frac{\pi}{2}\left(1 + 2\sum_{m=1}^{\infty} h^{m^2}\right)^2, \qquad h = e^{-\frac{\pi\Lambda'}{\Lambda}} = c^{-4n}.$$

The theorem is proved. ∎

The n-widths of periodic functions, which are represented as a convolution with some kernel $k \in$ NCVD, was studied by A. Pinkus [9] (see also [1]). In particular, it follows from [9], that for all $1 \le q \le \infty$

$$d_{2n}(A_H, L_q) = \lambda_{2n}(A_H, L_q) = d^{2n}(A_H, L_q) = \|K_H * h_n\|_q, \quad (18)$$

where $\|\cdot\|_q$ is the usual norm in the space $L_q := L_q[0, 2\pi]$. Since the function $K_h * h_n$ is found in direct form (see (14)) we can calculate the exact values of these n-widths.

THEOREM 5. *For all* $1 \le q < \infty$

$$\begin{aligned} d_{2n}(A_H, L_q) = \lambda_{2n}(A_H, L_q) = d^{2n}(A_H, L_q) &= \frac{4}{\pi}\left(\frac{2\pi}{\Lambda} I_{q1}(\lambda)\right)^{1/q} \\ &= \frac{8}{\pi}\left(2\sqrt{\pi}\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)}\right)^{1/q} e^{-Hn} + O(e^{-5Hn}), \end{aligned}$$

where

$$\lambda = 4e^{-2Hn}\left[\frac{\sum_{m=0}^{\infty} e^{-4Hnm(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-4Hnm^2}}\right]^2.$$

For $q = \infty$ we have from (14) and (18)

$$d_{2n}(A_H, L_\infty) = \lambda_{2n}(A_H, L_\infty) = d^{2n}(A_H, L_\infty) = \frac{4}{\pi}\arctan\sqrt{\lambda}.$$

Now from Lemma 1 and (17)

$$d_{2n}(A_H, L_\infty) = \lambda_{2n}(A_H, L_\infty) = d^{2n}(A_H, L_\infty)$$
$$= \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\frac{1}{\cosh[(2m+1)Hn]}.$$

These equations were previously calculated by V. M. Tikhomirov [11] (the complete proof was given by W. Forst [10]).

Let us consider some applications of Theorem 3 to optimal quadrature formulas. We are interested in the problem of approximate calculation of the integral

$$If := \int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}},$$

where $f \in W = BH_\infty(\mathscr{E}_c)$ or $Bh_\infty(\mathscr{E}_c)$, in terms of the values of f and its derivatives at a system of knots. Denote by

$$\tau_\alpha := \begin{pmatrix} x_1, \ldots, x_n \\ \alpha_1, \ldots, \alpha_n \end{pmatrix},$$

a system of distinct knots $x_1, \ldots, x_n \in [-1, 1]$ with multiplicities $\alpha_1, \ldots, \alpha_n$.

The error of the best quadrature formula for a given system τ_α is the number

$$e(\tau_\alpha, W) := \inf_{a_{jm}} \sup_{f \in W}\left| If - \sum_{j=1}^{n}\sum_{m=0}^{\alpha_j} a_{jm} f^{(m)}(x_j)\right|.$$

If $f \in Bh_\infty(\mathscr{E}_c)$ we mean by $f^{(m)}$ the partial derivative $\partial^m f/\partial x^m$. A quadrature formula is said to be best for a given system τ_α if it realizes the infimum.

Set

$$e(\alpha, W) := \inf_{-1 \le x_1 < \cdots < x_n \le 1} e(\tau_\alpha, W).$$

If the infimum is attained at the points $-1 \le x_1^0 < \cdots < x_n^0 \le 1$, then the best quadrature formula for this system of points, with multiplicities $\alpha = (\alpha_1, \dots, \alpha_n)$, is said to be optimal for the given α. The points $x_1^0, \dots, x_n^0$ are also called optimal.

It was proved in [2, 3] that

$$e(\alpha, BH_\infty(\mathscr{E}_c)) = \inf_{-\sqrt{k} < z_1 < \cdots < z_n < \sqrt{k}} \int_{-\sqrt{k}}^{\sqrt{k}} |B(z)|\, d\nu_k(z),$$

$$e(\alpha, Bh_\infty(\mathscr{E}_c)) = \frac{4}{\pi} \inf_{-\sqrt{k} < z_1 < \cdots < z_n < \sqrt{k}} \int_{-\sqrt{k}}^{\sqrt{k}} \arctan|B(z)|\, d\nu_k(z),$$

where

$$B(z) := \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - z_j z} \right)^{\mu_j}, \qquad \mu_j := 2[(\alpha_j + 1)/2]$$

(here the brackets denote the integral part) and k is determined by (8). Using Corollary 2 and the results of [2, 3] we obtain the following theorem.

THEOREM 6. *Let q be an even positive integer. Then for all $c > 1$:*

(i) *for all $q - 1 \le \alpha_j \le q$*

$$e(\alpha, BH_\infty(\mathscr{E}_c)) = \frac{\pi}{\Lambda} \lambda^{q/2} I_{q0}(\lambda) = 2^{q/2} \pi \frac{(q-1)!!}{(q/2)!} c^{-qn} + O(c^{-(q+4)n}),$$

$$e(\alpha, Bh_\infty(\mathscr{E}_c)) = \frac{4}{\Lambda} I_{q2}(\lambda) = 2^{q/2+2} \frac{(q-1)!!}{(q/2)!} c^{-qn} + O(c^{-(q+4)n}),$$

where λ is determined by (17), *and the unique system of optimal knots is the Chebyshev system* (9);

(ii) *for $\alpha_j \le 2$ the quadrature formulas*

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1 - x^2}} \approx \pi \frac{1 - \Delta_n(c)}{n} \sum_{j=1}^{n} f\left(\cos \frac{2j - 1}{2n} \pi \right),$$

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1 - x^2}} \approx \pi \frac{1 - \delta_n(c)}{n} \sum_{j=1}^{n} f\left(\cos \frac{2j - 1}{2n} \pi \right),$$

where

$$\Delta_n(c) := \frac{\lambda^2}{\Lambda} I_{40}(\lambda) = 6c^{-4n} + O(c^{-8n}),$$

$$\delta_n(c) := 2\frac{\lambda^2}{\Lambda}\int_0^1 \frac{t^4\,dt}{(1+\lambda^2 t^4)\sqrt{(1-t^2)(1-\lambda^2 t^2)}} = 12c^{-4n} + O(c^{-8n}),$$

are optimal for the classes $BH_\infty(\mathscr{E}_c)$ and $Bh_\infty(\mathscr{E}_c)$, respectively.

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